113 Class Problems: Direct Products and Sums

1. If two groups are isomorphic then both must satisfy exactly the same properties. With this in mind, prove the following:
   (a) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}$.
   (b) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \ncong \mathbb{Z}/9\mathbb{Z}$.

   Hint: Consider the order of elements on both sides.

   Solution:

a) $\text{ord}(([1]_3, [1]_5)) = 15$ and $|\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}| = 15$

$\Rightarrow$ $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ cyclic $\Rightarrow$ $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}$

b) $3([x]_3, [y]_3) = ([0]_3, [0]_3) \Rightarrow \text{Ord}(([x]_3, [y]_3)) \leq 3$

$\text{Ord}([1]_9) = 9 > 3$

$\Rightarrow$ $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3^2\mathbb{Z}$

2. Let G, K be groups. Prove the following:

$$G \times K \text{ Abelian} \iff G \text{ and } K \text{ Abelian}.$$

   Solutions:

($\Leftarrow$) Let $(g_1, k_1), (g_2, k_2) \in G \times K \Rightarrow (g_1, k_1) * (g_2, k_2) = (g_1 g_2, k_1 k_2)$

$\xrightarrow{G, K \text{ Abelian}} = (g_2 g_1, k_2 k_1) = (g_2, k_2) * (g_1, k_1)$

($\Rightarrow$) G, K isomorphic to subgroups of $G \times K$.

$G \times K$ Abelian $\Rightarrow$ All subgroups are Abelian $\Rightarrow$ G, K Abelian

3. A subgroup $H \subset G$ is proper if $H \neq G$.

Using question 2, prove that Sym_3 is **not** a direct sum of two proper subgroups.

Hint: Consider the sizes of proper subgroups.

Solutions:

$|Sym_3| = 6$. $H, K \subset Sym_3$ proper such that $Sym_3 = H \oplus K \cong H \times K$

$\Rightarrow$ $|H| = 2$, $|K| = 3$ without loss of generality

2, 3 prime $\Rightarrow$ $H \cong \mathbb{Z}/2\mathbb{Z}$, $K \cong \mathbb{Z}/3\mathbb{Z}$ $\Rightarrow$ $Sym_3 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

$\Rightarrow$ Sym_3 Abelian .

Contradiction, Sym_3 is non-Abelian.

4. Let $H_1, H_2 \subset G$ be subgroups such that $G = H_1 \oplus H_2$.

   (a) Prove that H_1 is a normal subgroup of G.

   (b) Prove that $G/H_1 \cong H_2$.

   Hint: Is there a natural surjective homomorphism from G to H_2?

   Solutions:

Define $\phi : G \longrightarrow H_2$

$g \longrightarrow h_2$

$\shortparallel$

$H_1 \ni h_1 * h_2 \in H_2$

Property 2 $\Rightarrow$ ϕ a homomorphism

$\text{Ker}\, \phi = H_1$ $\Rightarrow$ $H_1 \trianglelefteq G$

1st Isomorphism Theorem

$\text{Im}\, \phi = H_2$ $\Rightarrow$ $G/H_1 \cong H_2$